Five-Dimensional Unification of the Cosmological Constant and the Photon Mass

CHRISTOPHER KOHLER[1]

Institut für Theoretische und Angewandte Physik,
Universität Stuttgart, D-70550 Stuttgart, Germany

Abstract

Using a non-Riemannian geometry that is adapted to the 4+1 decomposition of space-time in Kaluza-Klein theory, the translational part of the connection form is related to the electromagnetic vector potential and a Stueckelberg scalar. The consideration of a five-dimensional gravitational action functional that shares the symmetries of the chosen geometry leads to a unification of the four dimensional cosmological term and a mass term for the vector potential.

1 Introduction

Recently, an alternative formulation of Kaluza-Klein theory [1, 2] has been presented which is based on a differential geometry that is adapted to the decomposition of the five-dimensional (5d) Kaluza-Klein space-time into 4d space-time and the internal S^1 manifold [3]. Physically, this approach leads to the same results as conventional Kaluza-Klein theory except that it allows for the Einstein-Cartan theory of 4d space-time.

In this article, the alternative formulation of Kaluza-Klein theory is investigated further. In particular, the translational connection on the 5d manifold is considered and it is shown that the part of it belonging to the fifth dimension is related to the electromagnetic vector potential and a Stueckelberg scalar. As a consequence, it will be seen that the $U(1)$ gauge symmetry of Maxwell theory can be interpreted as a translational symmetry of the internal space.

The use of the translational connection leads to a generalization of the alternative Kaluza-Klein theory in that a mass term for the electromagnetic field can be considered. The introduction of a 5d cosmological term that has the same symmetries as the geometry used leads, after dimensional reduction, to the Einstein-Cartan theory with cosmological constant coupled to massive Maxwell theory in the Stueckelberg formulation thus unifying the cosmological constant and the photon mass.

This article is organized as follows. In section 2, the alternative formulation of Kaluza-Klein theory is described in the language of fibre bundles. In section 3, the translational connection is introduced and its relationship with the electromagnetic vector potential is revealed. The 5d cosmological term and its dimensional reduction are considered in section 4. Section 5 contains some conclusions.

In this article, the following conventions are employed. Uppercase indices with values $0, 1, 2, 3, 5$ refer to 5d space-time where $A, B, \ldots$ are used for internal indices and $M, N, \ldots$ for coordinate indices. Lowercase indices with values $0, 1, 2, 3$ are used for 4d space-time with $a, b, \ldots$ denoting internal indices and $\mu, \nu, \ldots$ being coordinate indices. Explicit indices in parentheses will always be internal indices. The metric signature is $(- + + + (+))$. Throughout this article, geometrized units are used where $16\pi G = 1 = c$.

2 Five-Dimensional Unification of Einstein-Cartan Theory and Maxwell Theory

Kinematically, Kaluza-Klein theory is based on two assumptions: First, the 5d space-time manifold M^5 is decomposed into the 4d space-time manifold M^4 and a 1-sphere S^1 representing the internal space, that is, $M^5 = M^4 \times S^1$. Secondly, all geometrical fields defined on M^5 have a dependence only on the points of M^4. (We disregard massive excitations of the geometry.) Assuming a 5d local Lorentz gauge symmetry

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[1] E-mail: kohler@itap.physik.uni-stuttgart.de

on M^5, these two restrictions can be expressed as symmetry breakings: The 4+1 decomposition of M^5 is described by the symmetry breaking $SO(4,1) \to SO(3,1)$ on M^5 where $SO(3,1)$ represents the local Lorentz symmetry on M^4. The fact that the geometrical fields only depend on 4d spacetime corresponds to the symmetry breaking $SO(3,1) \to 1$ on each single internal manifold S^1.

Starting from a general Riemann-Cartan geometry on M^5, the consequent application of standard theorems for the reduction of linear connections [4] on the symmetry breakings leads to a unique geometry on M^5. This will be explained in this section. The reduction is described in some detail in order to be able to generalize it to affine connections in the following section.

A basis of tangent vectors $e_A = e_A^M \partial_M$ at a point of M^5 forms a linear frame. The set of all linear frames at all points of M^5 forms the bundle of linear frames $L(M^5)$. The restriction to orthonormal frames, which here means frames transforming under $SO(4,1)$, amounts to the choice of a 5d metric tensor γ_{MN} with respect to which the frames are orthonormal. The corresponding fibre bundle is the bundle of orthonormal frames $O(M^5)$. The generators of the Lie algebra $so(4,1)$ of $SO(4,1)$ will be denoted by J^{AB} and satisfy the Lie algebra

$$[J^{AB}, J^{CD}] = 2\eta^{C[A}J^{B]D} - 2\eta^{D[A}J^{B]C} \tag{1}$$

where η^{AB} is the 5d Minkowski metric and square brackets denote antisymmetrization. Each element $A = \frac{1}{2}A_{AB}J^{AB}$ of $so(4,1)$ induces a so called fundamental vector field A^* on $O(M^5)$ which is given by

$$A^* = A_{AB}e^{AM}\frac{\partial}{\partial e_B^M} \tag{2}$$

with $e^{AM} \equiv \eta^{AB}e_B^M$. The integral curves of A^* are 1-parameter groups of $SO(4,1)$ transformations on the fibres of $O(M^5)$ generated by A. A connection form ω on $O(M^5)$ is an $so(4,1)$ valued 1-form on $O(M^5)$ that satisfies $\omega(A^*) = A$ and transforms under global $SO(4,1)$ transformations of the frames according to the adjoint representation. From equation (2) then follows that ω is of the form

$$\omega = \frac{1}{2}\left(e_{MA}\mathrm{d}e_B^M + e_{MA}e_B^N\Gamma_{NP}^M\mathrm{d}x^P\right)J^{AB} \tag{3}$$

where the functions Γ_{NP}^M are the connection coefficients and $e_{AM} \equiv \eta_{AB}e_M^B$ with the coframe e^A which is determined by the duality relation $e^A(e_B) = \delta_B^A$. Under a local $SO(4,1)$ transformtion $e'_A = e_B\Omega^B{}_A$ of the frames, the connection form ω (or, more precisely, its pullback to M^5) transforms as

$$\omega^A{}_B \to \Omega^{-1A}{}_C\omega^C{}_D\Omega^D{}_B + \Omega^{-1A}{}_C\mathrm{d}\Omega^C{}_B. \tag{4}$$

The 4+1 decomposition of M^5 allows the use of special frames e_A whose basis vectors $e_{(5)}$ point in the fifth dimension, that is, $e_{(5)}$ is a multiple of ∂_5. These frames form a subbundle $Q(M^5)$ of $O(M^5)$ with structure group $SO(3,1)$. It is defined by the tensor

$$q_N^M = e_a^M e_N^a \tag{5}$$

which projects 5d vectors onto the 4d spacetime manifold M^4 and induces a metric on M^4. With respect to $O(M^5)$, this tensor can be regarded as a section of the associated fibre bundle with standard fibre $SO(4,1)/SO(3,1)$. The connection (3) is reducible to a connection ω' on $Q(M^5)$ if and only if q_N^M is parallel with respect to ω, from which follows

$$\omega_{a(5)M} = -e_a^N e_{(5)}^P D_M q_{NP} = 0 \tag{6}$$

where D_M denotes the covariant derivative corresponding to ω and $q_{MN} \equiv \gamma_{MP}q_N^P$.

The requirement to be able to perform a dimensional reduction means that geometrical quantities do not depend on the coordinate x^5 of S^1. This implies that a preferred basis e_a can be chosen where e_a is fixed along each internal space belonging to a point of M^4. Since each internal space S^1 at a point x^μ defines a subbundle $Q(S^1, x^\mu)$ of $Q(M^5)$ by restricting $Q(M^5)$ to the fibres over the S^1 manifold at the point x^μ of M^4, the preferred frames form a subbundle $S(S^1, x^\mu)$ of $Q(S^1, x^\mu)$ with structure group 1. The preferred frames $e_a(x^5, x^\mu)$ can be considered as a section of the associated fibre bundle over S^1 with

standard fibre $SO(3,1)$. The $SO(3,1)$ connection ω' induces a connection on each subbundle $Q(S^1, x^\mu)$ of $Q(M^4)$. These connections are reducible to connections on $S(S^1, x^\mu)$ if and only if $e_a(x^5, x^\mu)$ is parallel with respect to the induced connection, which means

$$\omega_{ab5} = e_a^M D_5 e_{bM} = 0 \tag{7}$$

The conditions (6) and (7) define a connection on M^5 that is adapted to the symmetries of the Kaluza-Klein reduction. It was previously called a semi-teleparallel connection [3] since it parallelizes the internal space while leaving the Riemann-Cartan connection on M^4 unrestricted.

We will base a five dimensional unification on this connection instead as on the Riemannian connection of conventional Kaluza-Klein theory. However, formally we will follow the same procedure in chosing an action functional as in Kaluza-Klein theory. To start with, we observe that similar to the unique decomposition

$$\omega^A{}_{BM} = \overset{\circ}{\omega}{}^A{}_{BM} - K^A{}_{BM} \tag{8}$$

of a Lorentz connection $\omega^A{}_{BM}$ into the Levi-Civita connection $\overset{\circ}{\omega}{}^A{}_{BM}$ and the contortion tensor $K^A{}_{BM}$, we have a unique decomposition [5]

$$\omega^A{}_{BM} = \tilde{\omega}^A{}_{BM} + S^A{}_{BM} \tag{9}$$

of a Lorentz connection $\omega^A{}_{BM}$ into a semi-teleparallel connection $\tilde{\omega}^A{}_{BM}$ and a tensor field $S^A{}_{BM}$ that is restricted by the condition that its projection onto M^4 vanishes,

$$q_Q^M q_R^N q_S^P S_{MNP} = 0 \tag{10}$$

where $S_{MNP} \equiv e_M^A e_N^B S_{ABP}$.

In order to obtain a unique action for the semi-teleparallel connection, we start with the Einstein-Cartan action on M^5,

$$S[e^A, \omega_{AB}] = \int_{M^5} \mathrm{d}^5 x \sqrt{-\gamma}\, R(e^A, \omega_{AB}). \tag{11}$$

Here, γ is the determinant of the 5d metric γ_{MN} and R is the 5d scalar curvature. We then insert the decomposition (9) for the Lorentz connection and determine the stationary point with respect to the tensor field $S^A{}_{BM}$. As a result, we obtain a functional of the semi-teleparallel connection $\tilde{\omega}$. In the case of the decomposition (8), this procedure leads to the Einstein-Hilbert action which is chosen as the action functional in conventional Kaluza-Klein theory. Thus, we expect by using the decomposition (9) to arrive at a sensible action functional for a 5d unification.

The determination of the stationary point of the action functional is best done using the preferred frames e_A that are adapted to the Kaluza-Klein geometry since in this basis we have $\omega_{a(5)M} = 0 = \omega_{ab5}$ and the condition (10) means $S_{abc} = 0$. Inserting the decomposition (9) into the Einstein-Cartan action and varying with respect to $S_{ab(5)}$ and $S_{(5)ab}$, we obtain

$$S_{ab(5)} = \tilde{K}_{ab(5)} \tag{12}$$
$$S_{(5)ab} = \tilde{K}_{(5)ab} \tag{13}$$

where $\tilde{K}_{ABC}$ is the contortion tensor of the semi-teleparallel connection $\tilde{\omega}_{ABC}$. The Einstein-Cartan action contains the components $S_{(5)a(5)}$ only within the linear term $S_{(5)a(5)}\tilde{K}^{ab}{}_b$. Hence, the action possesses a stationary point only if the 4d torsion is tracefree, $\tilde{K}^{ab}{}_b = 0$, which is the case in the absence of spinning matter. Reinserting (12) and (13) into the Einstein-Cartan action, we obtain the action

$$S[e^A, \tilde{\omega}_{AB}] = \int_{M^5} \mathrm{d}^5 x \sqrt{-\gamma}\left({}^4R - \frac{1}{4}T^{(5)ab}T_{(5)ab}\right). \tag{14}$$

Here, ${}^4R$ is the 4d scalar curvature and $T^A{}_{BC}$ is the 5d torsion tensor of $\tilde{\omega}$. We choose the action functional (14) as the action for the alternative five dimensional unification.

In order to make contact with electromagnetism, we parameterize the 5d coframe e^A. The fifth component of e_A is chosen to be

$$e_{(5)} = \mathrm{e}^{-\phi(x^\mu)}\partial_\theta \tag{15}$$

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where $\theta \equiv x^5$ is the coordinate along the internal S^1 and $\phi(x^\mu)$ is a scalar field as usual in the Kaluza-Klein theory. From the duality relation, the fifth component of the coframe then has the form

$$e^{(5)} = e^\phi \mathrm{d}\theta + C_\mu \mathrm{d}x^\mu \tag{16}$$

where C_μ is a 4d vector field. The remaining components of the basis and cobasis follow from equations (15) and (16) using the duality relation:

$$
\begin{aligned}
e_a &= e_a^\mu \partial_\mu - e_a^\mu C_\mu e^{-\phi} \partial_\theta \tag{17}\\
e^a &= e_\mu^a \mathrm{d}x^\mu. \tag{18}
\end{aligned}
$$

As in conventional Kaluza-Klein theory, the vector field C_μ shall be related to the electromagnetic vector potential A_μ. To find this relationship, we note that in Kaluza-Klein theory gauge transformations of A_μ are attributed to coordinate transformations along the internal space, that is, $\theta \to \theta + \lambda(x^\mu)$. Inserting this transformation into equation (16) leads to

$$e^{(5)} = e^\phi \mathrm{d}\theta + \left(e^\phi \partial_\mu \lambda + C_\mu\right) \mathrm{d}x^\mu. \tag{19}$$

From this follows that we have to choose

$$C_\mu = e^\phi A_\mu \tag{20}$$

in order to obtain the correct gauge transformation $A_\mu \to A_\mu + \partial_\mu \lambda$.

Using the chosen parameterization of the basis as the preferred frames for the semi-teleparallel connection, the fifth component of the torsion tensor is given by

$$T^{(5)}{}_{ab} = e^\phi e_a^\mu e_b^\nu F_{\mu\nu} \tag{21}$$

where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ represents the electromagnetic field strength. Then, the action functional (14) reads after dimensional reduction

$$S[e^a, \omega_{ab}, A_\mu, \phi] = \int_{M_4} \mathrm{d}^4 x \sqrt{-g} e^\phi \left({}^4R - \frac{1}{4}e^{2\phi} F_{\mu\nu} F^{\mu\nu}\right) \tag{22}$$

where g is the determinant of the 4d metric $g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab}$. Equation (22) coincides with the action of the Einstein-Cartan-Maxwell theory coupled to a scalar field.

3 Translational Connection

We next generalize the linear connection considered in the previous section to an affine connection, that is, we additionally introduce a translational connection on M^5 [4]. Applying symmetry breakings analogous to the ones used to define the semi-teleparallel connection, we will see that the translational connection has a natural relation with the electromagnetic vector potential.

The bundle of linear frames $L(M^5)$ can be extended to the bundle of affine frames $A(M^5)$ by generalizing linear frames e_A^M to affine frames (e_A^M, q^M) which include affine vectors $q^M = e_A^M q^A$ at each point of M^5. In the case of the bundle of orthonormal frames $O(M^5)$, the corresponding bundle of affine frames $P(M^5)$ consists of frames that transform under the 5d Poincaré group $P(5) = SO(4,1) \ltimes T^5$. $P(5)$ is generated by the generators (J^{AB}, P_A) which have, besides equation (1), the Lie brackets

$$
\begin{aligned}
[J^{AB}, P_C] &= 2\delta_C^{[B} \eta^{A]D} P_D \tag{23}\\
[P_A, P_B] &= 0. \tag{24}
\end{aligned}
$$

Under a Poincaré transformation, an affine frame transforms according to

$$\left(e_A^M, q^A\right) \to \left(\Omega^B{}_A e_B^M, \Omega^{-1A}{}_B(q^B - t^B)\right) \tag{25}$$

where t^A is a translation. The fundamental vector field A^* on $P(M^5)$ corresponding to the Lie algebra element $A = \frac{1}{2}A_{AB}J^{AB} + A^A P_A$ is

$$A^* = A_{AB} e^{AM} \frac{\partial}{\partial e_B^M} - A^A e_A^M \frac{\partial}{\partial q^M}. \tag{26}$$

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From this follows the connection form

$$\hat{\omega} = \omega + \varphi \tag{27}$$

with the translational connection form

$$\varphi = e_M^A \left(-\mathrm{d}q^M - \Gamma_{NP}^M q^N \mathrm{d}x^P + \Upsilon_N^M \mathrm{d}x^N \right) P_A \tag{28}$$

where ω and Γ_{NP}^M are the $SO(4,1)$ connection form and connection coefficients, respectively, defined in equation (3), and Υ_N^M are the connection coefficients of the translational connection. Under a local Poincaré transformation, φ^A transforms as

$$\varphi^A \to \Omega^{-1A}{}_B \left(\varphi^B + \mathrm{D}t^B \right) \tag{29}$$

where $\mathrm{D}t^A \equiv \mathrm{d}t^A + \omega^A{}_B t^B$. Since φ^A transforms under local Lorentz transformations as a tensor, we will set $\Upsilon_N^M = \delta_N^M$ in the following thus relating φ^A to the cobasis e^A. The translational connection form can then be written as

$$\varphi^A = -\mathrm{D}q^A + e^A. \tag{30}$$

Equation (30) was first introduced in reference [6]. We will follow reference [7] and refer to the bundle coordinates q^A as Poincaré coordinates.

We next apply the symmetry reductions described in the previous section to the translational connection. The first symmetry breaking $SO(4,1) \to SO(3,1)$ generalizes to

$$P(5) \to SO(3,1) \ltimes T^5. \tag{31}$$

The 5d translational symmetry is not broken since the 4+1 decomposition of space-time can be realized by splitting the translations into 4d space-time translations and internal ones. The reduced subbundle of $P(M^5)$ that corresponds to (31) will be denoted by $R(M^5)$. The translational part of $R(M^5)$ is not changed with respect to $P(M^5)$. Accordingly, the translational connection is not affected by the symmetry breaking (31). However, the second symmetry breaking, which is restricted to the fibres $R(S^1, x^\mu)$ of the bundle $R(M^5)$, generalizes to

$$SO(3,1) \ltimes T^5 \to 1 \times T^1 \tag{32}$$

where T^1 corresponds to the internal S^1 translations which are $U(1)$ transformations. The reason for (32) is that 4d space-time translations — like 4d space-time Lorentz transformations — must be independent of the coordinates of the internal space in order to be able to perform the dimensional reduction.

The induced connection form on $R(S^1, x^\mu)$ is

$$\hat{\omega} = \frac{1}{2} \omega_{ab5} \mathrm{d}\theta J^{ab} + \varphi_5^A \mathrm{d}\theta P_A. \tag{33}$$

The symmetry breaking (32) means that the connection (33) is reducible to the T^1 connection $\hat{\omega} = \varphi_5^{(5)} \mathrm{d}\theta P_5$ on each fibre. Consequently, besides $\omega_{ab5} = 0$, the components φ_5^a of the translational connection vanish.

Next we seek a physical interpretation of the semi-teleparallel translational connection. In accordance with the Kaluza-Klein theory, we require that the fields e_M^A and q^A only depend on the 4d space-time. (A dependence of $q^{(5)}$ on θ can be removed by a T^1 gauge transformation.) According to equation (30), the 4d space-time part of the coframe is given by

$$e_\mu^a = \partial_\mu q^a + \omega^a{}_{b\mu} q^b + \varphi_\mu^a. \tag{34}$$

This follows from the fact that $\omega^a{}_{(5)\mu}$ vanishes in a semi-teleparallel geometry. Equation (34) is the known relation between the 4d cobasis e_μ^a, the Riemann-Cartan connection $\omega^a{}_{b\mu}$, the 4d Poincaré coordinates q^a and the 4d translational connection φ_μ^a. Thus, equation (34) can be understood within a 4d Poincaré gauge theory [7].

For the components e_5^a of the coframe, we have from equation (30)

$$e_5^a = \partial_5 q^a + \omega^a{}_{B5} q^B + \varphi_5^a = 0 \tag{35}$$

because $\omega^a{}_{B5} = 0$ and $\varphi^a_5 = 0$ for a semi-teleparallel geometry and because q^a is independent of the coordinate θ of the internal space. Equation (35) is in accordance with our choice of the adapted frame (18).

The component $e^{(5)}_5$ is given by

$$e^{(5)}_5 = \partial_5 q^{(5)} + \omega^{(5)}{}_{a5} q^a + \varphi^{(5)}_5 = \varphi^{(5)}_5 \tag{36}$$

since $q^{(5)}$ does not depend on θ and $\omega^{(5)}{}_{a5} = 0$. Comparing equation (36) with equation (16), we find

$$\varphi^{(5)}_5 = e^\phi. \tag{37}$$

Finally, for the components $e^{(5)}_\mu$ follows

$$e^{(5)}_\mu = \partial_\mu q^{(5)} + \omega^{(5)}{}_{a\mu} q^a + \varphi^{(5)}_\mu = \partial_\mu q^{(5)} + \varphi^{(5)}_\mu. \tag{38}$$

Comparison with equation (16) yields

$$\varphi^{(5)}_\mu = C_\mu - \partial_\mu \sigma \tag{39}$$

where we have set $\sigma \equiv q^{(5)}$. In order to find the physical meaning of this equation, we consider a T^1 gauge transformation $\sigma \to \sigma - \lambda(x^\mu)$. From equations (29) and (39) follows

$$\varphi^{(5)}_\mu \to \varphi^{(5)}_\mu + \partial_\mu \lambda \tag{40}$$

which is a $U(1)$ gauge transformation. Hence, we can identify the translational connection $\varphi^{(5)}_\mu$ with the electromagnetic vector potential A_μ. Then, σ has the meaning of a Stueckelberg scalar [8]. Note that the identification

$$C_\mu = A_\mu + \partial_\mu \sigma \tag{41}$$

differs from (20) in that coordinate transformations of the internal space no longer induce $U(1)$ transformations — provided ϕ is not zero. Instead, the $U(1)$ gauge transformations are translational gauge transformations.

The construction of the action functional given in the previous section remains unchanged by the introduction of the translational connection. This follows from the fact that the coframe e^A is not affected by the symmetry reductions and there is no explicit dependence of the action functional on the translational connection.

To summarize this section, it has been shown that the electromagnetic vector potential A_μ can be interpreted as the fifth component $\varphi^{(5)}_\mu$ of a translational connection where $U(1)$ gauge transformations correspond to internal S^1 translations. Furthermore, the fifth component $q^{(5)}$ of the Poincaré coordinates can be interpreted as a Stueckelberg scalar σ.

It should be remarked that these identifications cannot be performed by generalizing a 5d Riemann-Cartan geometry or a 5d Riemannian geometry to include translations.

4 Cosmological Term

In this section, we consider the addition of a cosmological term to the action (22). In Kaluza-Klein theory, the cosmological term is

$$\Lambda \int_{M^5} \mathrm{d}^5 x \sqrt{-\gamma} \tag{42}$$

where Λ is the cosmological constant. Using differential forms, (42) may be rewritten as

$$\Lambda_{AB} \int_{M^5} e^A \wedge *e^B. \tag{43}$$

Here, e^A is a 5d orthonormal coframe, Λ_{AB} is a constant tensor, and $*$ is the duality operator defined by

$$\alpha \wedge *\beta = (\alpha, \beta) e \tag{44}$$

where α,β are p-forms, $(\cdot,\cdot)$ denotes the scalar product defined by the 5d metric induced by the coframe e^A, and e is the corresponding volume form. 5d Lorentz invariance requires the tensor Λ_{AB} to be of the form

$$\Lambda_{AB} = \frac{\Lambda}{5}\eta_{AB}. \tag{45}$$

Indeed, using equation (44) in (43) yields (42) where Λ_{AB} and Λ are related by equation (45).

The main difference between the approach to 5d unification presented in this work and traditional Kaluza-Klein theory is the breaking of the 5d Lorentz invariance to a 4d Lorentz invariance. Therefore, we have also to break down the 5d Lorentz invariance of the cosmological term. This can be done by using a different duality operator in (43). We can define a new duality operator $\sharp$ by

$$e^A \wedge \sharp e^B = e_M^A e_N^B \, {}^\sharp\gamma^{MN} \sqrt{-{}^\sharp\gamma} \, \mathrm{d}^5 x \tag{46}$$

where ${}^\sharp\gamma^{MN}$ is a 5d metric different from γ^{MN}. Since we want to retain 4d Lorentz invariance, equation (46) has to be invariant under 4d Lorentz transformations which means that the 4d part of ${}^\sharp\gamma^{MN}$ is the one of γ^{MN}, that is, ${}^\sharp\gamma^{\mu\nu} = g^{\mu\nu}$. For ${}^\sharp\gamma^{MN}$ we use the parameterization

$$ {}^\sharp\gamma^{MN} = \begin{pmatrix} g^{\mu\nu} & -B^\nu \\ -B^\mu & \mathrm{e}^{-2\psi} + B^\mu B_\mu \end{pmatrix} \tag{47}$$

where B^μ is a 4d vector field and ψ is a scalar field.

The restricted invariance under 4d Lorentz transformations requires Λ_{AB} to be of the form

$$\Lambda_{AB} = \begin{pmatrix} \frac{\Lambda}{4}\eta_{ab} & 0 \\ 0 & \Sigma \end{pmatrix} \tag{48}$$

where Σ is a constant. Using this equation and the modified duality operator $\sharp$, (43) yields after dimensional reduction

$$\int_{M^4} \mathrm{d}^4 x \sqrt{-g} \mathrm{e}^\psi \Big[\Lambda_{ab} e_\mu^a e_\nu^b g^{\mu\nu}$$
$$+ \quad \Sigma e_\mu^{(5)} e_\nu^{(5)} g^{\mu\nu} - 2\Sigma B^\mu e_\mu^{(5)} e_5^{(5)} + \Sigma e_5^{(5)} e_5^{(5)} \left(\mathrm{e}^{-2\psi} + B^\mu B_\mu \right) \Big]. \tag{49}$$

If we insert the $e^{(5)}$ given by equations (16) and (41), and use that $g^{\mu\nu} = e_a^\mu e_b^\nu \eta^{ab}$, we obtain

$$\int_{M^4} \mathrm{d}^4 x \sqrt{-g} \mathrm{e}^\psi \Big[\Lambda + \Sigma \mathrm{e}^{2\phi-2\psi}$$
$$+ \quad \Sigma \left(A_\mu + \partial_\mu\sigma - \mathrm{e}^\phi B_\mu \right) \left(A_\nu + \partial_\nu\sigma - \mathrm{e}^\phi B_\nu \right) g^{\mu\nu} \Big]. \tag{50}$$

The expression (50) contains a mass term for the vector potential A_μ. This is particularly clear in the case $\psi = \phi$ and $B_\mu = 0$ for which (50) reads

$$\int_{M^4} \mathrm{d}^4 x \sqrt{-g} \mathrm{e}^\phi \left[\Lambda + \Sigma + \Sigma \left(A_\mu + \partial_\mu\sigma \right) \left(A_\nu + \partial_\nu\sigma \right) g^{\mu\nu} \right]. \tag{51}$$

The functional (51) consists of two parts, a 4d cosmological term with the cosmological constant $\Lambda + \Sigma$ and a mass term for the vector potential in the Stueckelberg formalism where the photon mass is given by

$$m = \hbar\sqrt{\Sigma}. \tag{52}$$

Other choices of the fields B_μ and ψ are possible. B_μ may be interpreted as an external current.

In summary, we see that from the viewpoint of a 5d unification, the cosmological constant and the photon mass are of the same origin.

5   Conclusions

The main achievement of Kaluza-Klein theory is the unification of 4d gravitational concepts and notions from Maxwell theory into 5d gravitational concepts. This has so far been done, e.g., for the 4d metric and the vector potential which are unified in the 5d metric, and for the momentum of a point particle and its charge which are unified in a 5d momentum. In this article, this Kaluza-Klein programme has been extended in two ways. First, the 4d translational gauge symmetry of gravitation and the $U(1)$ gauge symmetry of Maxwell theory are unified in a 5d translational symmetry. Secondly, it has been shown that the cosmological constant and the photon mass can be unified in a 5d cosmological tensor.

Whether the latter unification is of a deeper significance, cannot be shown by the classical methods used in this paper. However, a further test of the usefulness of the approach would be the generalization to nonabelian gauge theories. Heuristically, if we consider the gauge theory of electroweak interactions, the minimal choice for the internal manifold in a higher dimensional unification is the homogeneous space $U(1) \times SU(2)/U(1)$. Since this space is three-dimensional, the incorporation of a cosmological term along the lines of section 4 of this article would lead to three mass terms for the four gauge bosons, which qualitatively agrees with the mass spectrum of the gauge bosons of electroweak interactions.

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